Maxwell’s equations

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Maxwell

(13 June 1831 – 5 November 1879) was a Scottish physicist.

Famous equations published in 1861

$\partial\Omega$

Maxwell’s equations: integral form

$$\oiint \mathbf{E}\cdot \mathrm{d}\mathbf{S} = \frac{Q(V)}{\varepsilon_0}$$

Gauss's law

$$\oiint \mathbf{B}\cdot \mathrm{d}\mathbf{S} = 0$$

Gauss's law for magnetism: no magnetic monopole!

$$\oint_{\partial\Sigma} \mathrm{B}\cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 I + \mu_0\varepsilon_0 \iint_{\Sigma} \frac{\partial \mathbf{E}}{\partial t}\cdot \mathrm{d}\mathrm{S}$$

Ampère's law (with Maxwell's addition)

$$\oint_{\partial\Sigma} \mathbf{E}\cdot \mathrm{d}\boldsymbol{\ell} = -\iint_{\Sigma} \frac{\partial \mathbf{B}}{\partial t}\cdot \mathrm{d}\mathbf{S}$$

Faraday's law of induction (Maxwell–Faraday equation)

Relation of the speed of light and electric and magnetic vacuum constants

$$\begin{aligned} c &= 1/\sqrt{\epsilon_0\mu_0} = 299792458 \text{ m/s} \\ \epsilon_0 &\approx 8.854 \cdot 10^{-12} \text{ As/Vm}, \\ \mu_0 &= 4\pi \cdot 10^{-7} \text{ Vs/Am}, \end{aligned}$$

ε_0	permittivity of free space, also called the electric constant	As/Vm or F/m (farad per meter)
μ_0	permeability of free space, also called the magnetic constant	Vs/Am or H/m (henry per meter)

Differential operators

$\nabla \cdot$ the divergence operator $\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

div

$\nabla \times$ the curl operator $\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$

curl, rot

Other notation used

$$\partial_x = \frac{\partial}{\partial x}$$

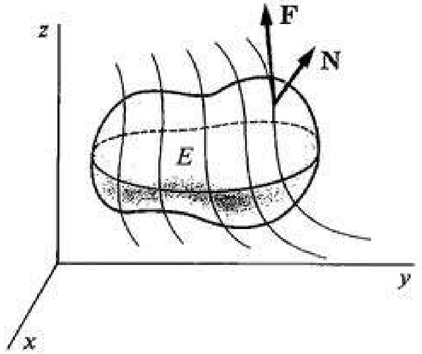
$\frac{\partial}{\partial t}$ the partial derivative with respect to time

Transition from integral to differential form

Gauss’ theorem for a vector field **F**(**r**)

$$\iiint_V (\nabla \cdot \mathbf{F})\, dV = \mathop{\oiint}_{\mathbf{S}} (\mathbf{F} \cdot \mathbf{n})\, dS.$$

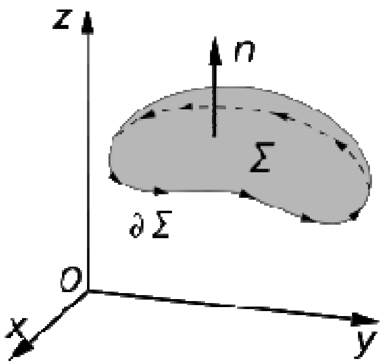
Volume V, surrounded by surface



Stokes' theorem for a vector field **F**(**r**)

$$\iint_\Sigma \nabla \times \mathbf{F} \cdot d\mathbf{\Sigma} = \oint_{\partial\Sigma} \mathbf{F} \cdot d\mathbf{r},$$

Surface Σ, surrounded by contour



$\partial\Omega$

Maxwell’s equations: integral form

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Ampère's law
(with Maxwell's addition)

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Maxwell–Faraday equation
(Faraday's law of induction)

Maxwell's equations (SI units) differential form

$$\nabla . \mathbf{E} = \frac{\rho}{\epsilon_0}$$

ρ density of charges

$$\nabla . \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \times \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j}$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c^2 \epsilon_0} \mathbf{j}$$

j density of current

$$c^2 = \frac{1}{\mu_0 \epsilon_0}$$

Electric and magnetic fields and units

E	electric field,	volt per meter, V/m
B	the magnetic field or magnetic induction	tesla, T
D	electric displacement field	coulombs per square meter, C/m^2
H	magnetic field	ampere per meter, A/m

Constitutive relations

These equations specify the response of bound charge and current to the applied fields and are called constitutive relations.

$$\mathbf{D}(\mathbf{r},t) = \varepsilon_0\mathbf{E}(\mathbf{r},t) + \mathbf{P}(\mathbf{r},t)$$

P is the polarization field,

$$\mathbf{H}(\mathbf{r},t) = \frac{1}{\mu_0}\mathbf{B}(\mathbf{r},t) - \mathbf{M}(\mathbf{r},t),$$

M is the magnetization field, then

$$\mathbf{D} = \varepsilon\mathbf{E}\,, \quad \mathbf{H} = \mathbf{B}/\mu$$

where ε is the permittivity and μ the permeability of the material.

× ×

Wave equation

∇

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \overset{0}{\cancel{\frac{1}{c^2 \epsilon_0}\mathbf{j}}}$$

$$\nabla \times (\nabla \times \vec{B}) = \nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = -\nabla^2 \vec{B}$$

$$\nabla \times \frac{\partial}{\partial t}(\vec{E}) = \frac{\partial}{\partial t}(\nabla \times \vec{E}) = \frac{\partial}{\partial t}(-\frac{1}{c^2}\frac{\partial}{\partial t}\vec{B}) = -\frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\nabla^2 \vec{B} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

Double vector product rule is used

a x **b** x **c** = (**a**·**c**) **b** - (**a**·**b**) **c**

2 more differential operators

∇^2 or Δ **Laplace operator or Laplacian**

$$\nabla^2 \vec{A} = \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}$$

d'Alembert operator or **d'Alembertian**

$$\lrcorner = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta.$$

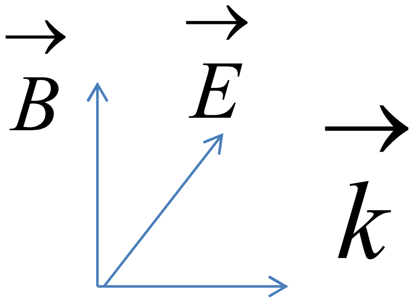
Plane waves

Thus, we seek the solutions of the form:

$$\vec{B} = \vec{B}_0 \, Exp[i(\vec{k} \cdot \vec{r} - \omega t)]$$

$$\vec{E} = \vec{E}_0 \, Exp[i(\vec{k} \cdot \vec{r} - \omega t)]$$

From Maxwell's equations one can see that



$$\nabla \times \vec{B} = i \, \vec{k} \times \vec{B}$$ is parallel to $\vec{E}$

$$\nabla \times \vec{E} = i \, \vec{k} \times \vec{E}$$ is parallel to $\vec{B}$

Energy transfer and Pointing vector

Differential form of Pointing theorem

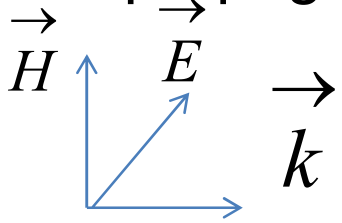
$$-\frac{\partial u}{\partial t} = \nabla \cdot \mathbf{S} + \mathbf{J} \cdot \mathbf{E},$$

$$u = \frac{1}{2}\left(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}\right). \qquad u = \frac{1}{2}\left(\varepsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0}\mathbf{B}^2\right)$$

u is the density of electromagnetic energy of the field

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \parallel \mathbf{n} = \frac{\mathbf{k}}{k}$$

S is directed along the propagation direction



Integral form of Pointing theorem

$$-\frac{\partial}{\partial t}\int_V u\,dV = \oiint \mathbf{S} \cdot d\mathbf{A} + \int_V \mathbf{J} \cdot \mathbf{E}\,dV$$

Energy quantities continued

$$\vec{B} = \vec{B}_{\max} exp[i(\vec{k} \cdot \vec{r} - \omega t)]$$

$$\vec{E} = \vec{E}_{\max} exp[i(\vec{k} \cdot \vec{r} - \omega t)]$$

$$B_{\max} = E_{\max} / c$$

Observable are real values:

$$\vec{B} = \vec{B}_{\max} \cos[i(\vec{k} \cdot \vec{r} - \omega t + \Delta\varphi)]$$

$$\vec{E} = \vec{E}_{\max} \cos[i(\vec{k} \cdot \vec{r} - \omega t)]$$

$$I = S_{av} = E_{\max}{}^2 / 2c\mu_0 = cB_{\max}{}^2 / 2\mu_0 = cu_{av}$$

$$u_{av} = E_{\max}{}^2 / (4c^2\mu_0) + B_{\max}{}^2 / 4\mu_0$$